ON THE THEORY OF IDEALLY PLASTIC ANISOTROPY

(K TEORII IDEAL'NOI PLASTICHESKOI ANIZOTROPII)

PMM Vol.23, No.6, 1959, pp. 1107-1114

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(Received 11 July 1959)

The behavior of an ideally rigid plastic body is investigated under a generalized Tresca plasticity condition.

We remark that Hill [1] was apparently the first to assume a plasticity condition for an ideal anisotropic body: a generalized Mises plasticity condition which has been employed repeatedly in applications.

1. We consider an element of the medium in an orthogonal coordinate system x, y, z. We shall suppose that the flow limits are known for tension and compression of the element in any direction in the x, y, z system.

Thus, if the flow limits for tension are denoted by k and those for compression by s, then in the general case we shall have $k = k(l_i)$, $s = s(l_i)$, $i = 1, 2, 3, \ldots$, where the l_i are direction cosines specifying the direction of the tension or compression relative to the x, y, z axes. The relation

$$l_1^2 + l_2^2 + l_3^2 = 1 \qquad (1.1)$$

holds.

By projecting line segments in the directions of tension or compression proportional to the values of k and s, we obtain surfaces which we call the surfaces of anisotropy in tension and compression. For an isotropic body these surfaces are evidently spheres. In the general case the surfaces of anisotropy in tension and compression may be discontinuous, as for example, in layered materials.

By 1, 2, 3 we denote three orthogonal directions along which the principal stresses σ_1, σ_2, σ_3 act. The orientation of the 1, 2, 3 axes with respect to the x, y, z coordinate system is determined by the table of direction cosines.

We consider the plasticity condition. By virtue of the supposition that the plasticity conditions are independent of the hydrostatic pressure, the plasticity conditions are interpreted in a space of principal stresses as a certain cylindrical surface whose generators are parallel to $\sigma_1 = \sigma_2 = \sigma_3$. The curve of intersection of the plasticity condition with the plane $\sigma_1 + \sigma_2 + \sigma_3 = 0$ we call the curve of plasticity. Figure 1 shows the curve of plasticity determining the extremal plastic flow properties of the medium which are possible under non-concave conditions of plasticity, passing through the known points fixed by the flow limits for tension or compression.

	1	2	3
x	l_1	m_1	n_1
	l_2	m	n_2
z	l_3	m_3	n_3

We must note that the condition of non-concavity of the curve shown in Fig. 1 establishes certain evident limits on the values of the limits for tension and compression.

Thus, with these assumptions, a certain six-sided prism with boundaries parallel to $\sigma_1 = \sigma_2 = \sigma_3$, completely determined by the flow limits for tension and compression for each position of the axes 1, 2, 3 may be interpreted as giving the conditions of plasticity for an ideally plastic anisotropic body. Changing the orientation of the 1, 2, 3 axes in the x, y, z system changes the prism specifying the plasticity conditions.

We shall write the plasticity conditions. We denote the flow limits in tension by $k_1(l_i)$, $k_2(m_i)$, $k_3(n_i)$, and the flow limits in compression by $s_1(l_i)$, $s_2(m_i)$, $s_3(n_i)$. We write the equation of the plane, parallel to $\sigma_1 = \sigma_2 = \sigma_3$ in the form

$$a\sigma_1 + b\sigma_2 + c\sigma_3 = 0 \qquad (a + b + c = 0) \tag{1.3}$$

The required plasticity conditions are easily obtained from this as (Fig. 1)

(1.4)

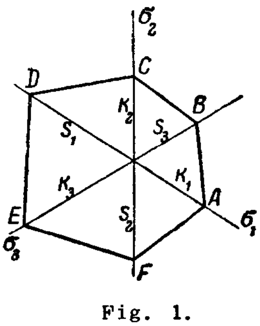

Fig. 1.

$$\frac{\sigma_1}{k_1} - \left(\frac{1}{k_1} - \frac{1}{s_3}\right)\sigma_2 - \frac{\sigma_3}{s_3} = 1 \qquad (AB)$$

$$-\left(\frac{1}{k_2} - \frac{1}{s_3}\right)\sigma_1 + \frac{\sigma_2}{k_2} - \frac{\sigma_3}{s_3} = 1 \qquad (BC)$$

$$-\frac{\sigma_1}{s_1} + \frac{\sigma_2}{k_2} + \left(\frac{1}{s_1} - \frac{1}{k_2}\right)\sigma_3 = 1 \qquad (CD)$$

$$-\frac{\sigma_1}{s_1} + \left(\frac{1}{s_1} - \frac{1}{k_3}\right)\sigma_2 + \frac{\sigma_3}{k_3} = 1 \qquad (DE)$$

$$\left(\frac{1}{s_2} - \frac{1}{k_3}\right)\sigma_1 - \frac{\sigma_2}{s_2} + \frac{\sigma_3}{k_3} = 1 \qquad (EF)$$

$$\frac{\sigma_1}{k_1} - \frac{\sigma_2}{s_2} - \left(\frac{1}{k_1} - \frac{1}{s_2}\right)\sigma_3 = 1 \qquad (FA)$$

We determine the equations for the edges of the prism, considered as plasticity conditions. It is easy to convince oneself that the required equations may be written in the form

$$\sigma_i = \sigma_j = \sigma_k - 2k_k, \qquad \sigma_i = \sigma_j = -\sigma_k - 2s_k \tag{1.5}$$

The condition of complete plasticity for an isotropic body may be generalized by relations (1.5). In the following we consider the condition $\sigma_1 = \sigma_2 = \sigma_3 - 2k_3$, in which we omit the index 3 in k_3.

It is well-known that

$$\begin{aligned} \sigma_x &= \sigma_1 l_1^2 + \sigma_2 m_1^2 + \sigma_3 n_1^2 \\ \tau_{xy} &= \sigma_1 l_1 l_2 + \sigma_2 m_1 m_2 + \sigma_3 n_1 n_2 \end{aligned} \qquad \begin{pmatrix} x & y & z \\ 1 & 2 & 3 \end{pmatrix} \tag{1.6}$$

Here and in the following the symbols (xyz), $(1\ 2\ 3)$ denote that the remaining relations are to be obtained by a regular permutation of indices. By making use of (1.6) we obtain

$$\sigma_x = \sigma - \frac{2}{3} k + 2kn_1^2, \qquad \tau_{xy} = 2kn_1 n_2 \qquad (xyz)\ (1, 2, 3) \tag{1.7}$$

where

$$k = k(n_i), \qquad \sigma = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z), \qquad n_1 = \cos\theta_1, \qquad n_2 = \cos\theta_2, \qquad n_3 = \cos\theta_3$$

From (1.7) it is easy to obtain the relation

$$\left(\sigma_x - \sigma - \frac{2}{3} k\right)\left(\sigma_y - \sigma - \frac{2}{3} k\right) = \tau_{xy}^2 \qquad (xyz) \tag{1.8}$$

or

$$\left(\sigma_x - \sigma - \frac{2}{3} k\right)\tau_{yz} - \tau_{zx}\tau_{xy} = 0 \qquad (xyz) \tag{1.9}$$

Expressions for the plasticity conditions corresponding to the prism boundaries may be found analogously as generalizations of those given by Levi [2].

We substitute the relations (1.7) into the equation of equilibrium

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \qquad (xyz) \tag{1.10}$$

We obtain

$$\begin{aligned} &\frac{1}{2}\frac{\partial p}{\partial x} + \left[\frac{\partial k}{\partial n_1}\left(\frac{1}{3} - n_1^2\right) - 2kn_1\right]\sin\theta_1 \frac{\partial \theta_1}{\partial x} - \left[\frac{\partial k}{\partial n_1} n_1 n_2 + kn_2\right]\sin\theta_1 \frac{\partial \theta_1}{\partial y} - \\ &\quad - \left[\frac{\partial k}{\partial n_1} n_1 n_3 + kn_3\right]\sin\theta_1 \frac{\partial \theta_1}{\partial z} + \frac{\partial k}{\partial n_2}\left(\frac{1}{3} - n_1^2\right)\sin\theta_2 \frac{\partial \theta_2}{\partial x} - \\ &- \left[\frac{\partial k}{\partial n_2} n_1 n_2 + kn_1\right]\sin\theta_2 \frac{\partial \theta_2}{\partial y} - \frac{\partial k}{\partial n_2} n_1 n_3 \sin\theta_2 \frac{\partial \theta_2}{\partial z} + \frac{\partial k}{\partial n_3}\left(\frac{1}{3} - n_1^2\right)\sin\theta_3 \frac{\partial \theta_3}{\partial x} - \\ &\quad - \frac{\partial k}{\partial n_3} n_1 n_2 \sin\theta_3 \frac{\partial \theta_3}{\partial y} - \left[\frac{\partial k}{\partial n_3} n_1 n_2 + kn_1\right]\sin\theta_3 \frac{\partial \theta_3}{\partial z} = 0 \end{aligned} \tag{1.11}$$

in which

$$p = \sigma, \qquad \cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 = 1 \tag{1.12}$$

The system of four equations (1.11), (1.12) in the four unknowns p, θ_i, is a system of hyperbolic type. Upon writing the equation of the characteristic surface in the form $\Psi(x, y, z)$, we find that the characteristic determinant in vector form is

$$-(\mathrm{grad}\,\Psi)^2 + 2(\mathrm{grad}\,\Psi\cdot\mathbf{n})^2 - \frac{1}{k}(\mathrm{grad}\,\Psi\cdot\mathbf{n}) \tag{1.13}$$

$$(\mathrm{grad}\,\Psi\cdot\mathbf{a}) + \frac{1}{k}(\mathrm{grad}\,\Psi\cdot\mathbf{n})(\mathbf{a}\cdot\mathbf{n}) = 0$$

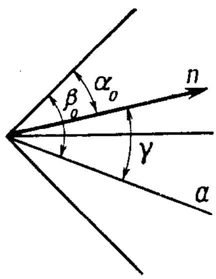

Fig. 2.

where

$$\mathbf{n} = \mathbf{n}\{n_i\}, \qquad \mathbf{a} = \mathbf{a}\left\{\frac{\partial k}{\partial n_i}\right\}$$

We denote by α the angle between the vectors grad Ψ and $\mathbf{n}$, by β the angle between the vectors grad ψ and $\mathbf{a}$, and by γ the angle between the vectors $\mathbf{a}$ and $\mathbf{n}$. We obtain from (1.3)

$$-1 + 2\cos^2\alpha - b\cos\alpha\cos\beta + b\cos^2\alpha\cos\gamma = 0 \qquad \left(b = \frac{|\mathbf{a}|}{k}\right) \tag{1.14}$$

At a given point in the body under a given state of stress the directions of $\mathbf{a}$ and $\mathbf{n}$ are fixed and the direction of grad Ψ describes a certain characteristic cone. Consequently, the angle γ is fixed and the angles α and β determine the directions of the generators of the characteristic cone.

We suppose that the directions of $\mathbf{a}$, $\mathbf{n}$ and grad Ψ lie in one plane (Fig. 2). Then $\alpha_0 = \beta_0 + \gamma_0$. We obtain from (1.4)

$$\mathrm{ctg}\, 2\alpha_0 = \frac{1}{2} b \sin\gamma \tag{1.15}$$

It follows from (1.15) that the directions of grad Ψ in the **an** plane are orthogonal. We trace lines in this plane which make an angle equal to $\pi/4$ with the directions of grad ψ. These directions, which we call the axes of the characteristic cone, make angles with the direction of $\mathbf{n}$ equal to $\alpha_0 - \pi/4$. We find that these are equal to the angles formed by the generators of any characteristic cone with the axis. It is easy to show that the angle sought is determined from the expression

$$\cos\nu = \frac{\sqrt{2}}{2}\left(\frac{\cos^2\alpha + \sin\alpha_0\cos\alpha_0}{\cos\alpha'(\sin\alpha_0 + \cos\alpha_0)}\right)$$

Evidently for an isotropic material $\mathbf{a} = 0$, $\alpha = \alpha_0 = \pi/4$, $\cos\nu = 1/2\sqrt{2}$.

In the general case the characteristic cone will not be a circular one. The circumstance is a result of the material anisotropy.

We remark that $\cos a \neq 0$ follows from (1.15).

We return to the determination of the law of plastic flow. We find, upon employing the Mises relation concerning the extremum of the plastic work [3], as well as the corresponding generalizations of Koiter [4] and Prager [5] for the case of angular points on the flow surface, that from the plasticity conditions (1.8) or (1.9)

$$\varepsilon_x = \frac{2}{3}\lambda_1\left(1-\frac{\partial k}{\partial\sigma_x}\tau_{yz}\right)-\frac{2}{3}\lambda_2\left(2-\frac{\partial k}{\partial\sigma_x}\tau_{zx}\right)-\frac{2}{3}\lambda_3\left(2-\frac{\partial k}{\partial\sigma_x}\tau_{xy}\right)$$

$$\varepsilon_{xy} = -\lambda_1\left(\frac{2}{3}\frac{\partial k}{\partial\tau_{xy}}\tau_{yz}+\tau_{xz}\right)- \qquad \binom{x\ y\ z}{1\ 2\ 3} \tag{1.16}$$

$$-\lambda_2\left(\frac{2}{3}\frac{\partial k}{\partial\tau_{xy}}\tau_{xz}+\tau_{yz}\right)+\lambda_3\left[\left(\sigma_z-\sigma-\frac{2}{3}k\right)-\frac{2}{3}\frac{\partial k}{\partial\tau_{xy}}\tau_{xy}\right]$$

where

$$\frac{\partial k}{\partial\sigma_{ij}}=\frac{\partial k}{\partial n_1}\frac{\partial n_1}{\partial\sigma_{ij}}+\frac{\partial k}{\partial n_2}\frac{\partial n_2}{\partial\sigma_{ij}}+\frac{\partial k}{\partial n_3}\frac{\partial n_3}{\partial\sigma_{ij}}$$

In a given case the necessity of differentiating the function k with respect to the stress components is apparent.

Further, it is necessary to deduce certain relations. We obtain from (1.7)

$$2\sigma_x-\sigma_y-\sigma = 2k(3n_1^2-1), \qquad \tau_{xy}^2+\tau_{xz}^2 = 4k^2n_1^2(1-n_1^2) \qquad \binom{x\ y\ z}{1\ 2\ 3} \tag{1.17}$$

It follows from (1.17) that

$$\frac{(2\sigma_x-\sigma_y-\sigma_z)^2}{\tau_{xy}^2+\tau_{xz}^2}=\frac{(3n_1^2-1)^2}{n_1^2(1-n_1^2)} \qquad \binom{x\ y\ z}{1\ 2\ 3} \tag{1.18}$$

and from (1.18)

$$\frac{\partial n_1}{\partial\sigma_x}=\frac{n_1(1-n_1^2)}{1+n_1^2}, \qquad \frac{\partial n_1}{\partial\sigma_y}=\frac{\partial n_1}{\partial\sigma_z}=-\frac{n_1(1-n_1^2)}{2(1+n_1^2)} \qquad \binom{x\ y\ z}{1\ 2\ 3}$$

$$\frac{\partial n_1}{\partial\tau_{xy}}=-\frac{n_2(3n_1^2-1)}{2(1+n_1^2)}, \qquad \frac{\partial n_1}{\partial\tau_{xz}}=-\frac{n_3(3n_1^2-1)}{2(1+n_1^2)}, \qquad \frac{\partial n_1}{\partial\tau_{yz}}=0 \tag{1.19}$$

By substitution of expressions (1.19) into relations (1.16) and eliminating the quantities λ_1, λ_2, λ_3 we obtain

$$d_1\left[\varepsilon_x+\varepsilon_{xy}\frac{\sigma_y-\sigma-{}^2/_3k}{\tau_{xy}}+\varepsilon_{xz}\frac{\sigma_z-\sigma-{}^2/_3k}{\tau_{xz}}\right]=$$

$$=d_2\left[\varepsilon_{xy}\frac{\sigma_x-\sigma-{}^2/_3k}{\tau_{xy}}+\varepsilon_y+\varepsilon_{yz}\frac{\sigma_z-\sigma-{}^2/_3k}{\tau_{yz}}\right]= \tag{1.20}$$

$$=d_3\left[\varepsilon_{xz}\frac{\sigma_x-\sigma-{}^2/_3k}{\tau_{xz}}+\varepsilon_{yz}\frac{\sigma_y-\sigma-{}^2/_3k}{\tau_{yz}}+\varepsilon_z\right]$$

where

$$d_i = 2k + \frac{1}{n_i}\frac{\partial k}{\partial n_i} - \left(\frac{\partial k}{\partial n_1}n_1 + \frac{\partial k}{\partial n_2}n_2 + \frac{\partial k}{\partial n_3}n_3\right)$$

It is easy to see that the incompressibility condition

$$\varepsilon_x + \varepsilon_y + \varepsilon_z = 0 \tag{1.21}$$

is fulfilled.

The three equations (1.20), (1.21) in the three unknown velocity components of the displacements u, v, w belong to a hyperbolic type, of which the characteristic surfaces are determined according to equation (1.13). Thus, in the case considered, the system of equations characterizing the anisotropy and the system determining the states of stress and strain coincide.

Although the states of stress and strain corresponding to the plasticity conditions at the prism boundaries have been touched upon, this question must be considered separately.

2. We consider the twisting of a prismatic rod of ideally rigid plastic anisotropic material. We suppose that the rod anisotropy is such that it is independent of z, the axial coordinate. As usual, we set

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0 \tag{2.1}$$

We satisfy relations (1.6), setting $\sigma_1 = -\sigma_2$, $\sigma_3 = 0$, for which the orientation of the 1, 2, 3 axes in the x, y, z system is determined according to the table of direction cosines.

TABLE

	1	2	3
x	$\frac{\sqrt{2}}{2}\cos\theta$	$\frac{\sqrt{2}}{2}\cos\theta$	$-\sin\theta$
y	$\frac{\sqrt{2}}{2}\sin\theta$	$\frac{\sqrt{2}}{2}\sin\theta$	$-\cos\theta$
z	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	0

Any of the plasticity conditions (1.4) leads to the expression

$$\sigma_1 = \sigma_1(\theta) \quad \text{or} \quad \tau_{xz}^2 + \tau_{yz}^2 = k^2(\theta) \tag{2.2}$$

By substitution of the expressions for stress $\tau_{xz} = k(\theta)\cos\theta$, $\tau_{yz} = k(\theta)\sin\theta$ into the single equation of equilibrium

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0$$

we obtain

$$\left(-k\sin\theta + \frac{dk}{d\theta}\cos\theta\right)\frac{\partial\theta}{\partial x} + \left(k\cos\theta + \frac{dk}{d\theta}\sin\theta\right)\frac{\partial\theta}{\partial y} = 0 \tag{2.3}$$

The characteristics of equation (2.3) are determined from

$$\frac{dx}{-k\sin\theta + k'\cos\theta} = \frac{dy}{k\cos\theta + k'\sin\theta} = \frac{d\theta}{0} \qquad \left(k' = \frac{dk}{d\theta}\right) \tag{2.4}$$

It follows from (2.4) that the characteristics are the straight lines

$$y = \frac{k\cos\theta + k'\sin\theta}{-k\sin\theta + k'\cos\theta}\, x + \Phi(\theta) \tag{2.5}$$

along which θ = const. Further, by making use of the associated law of flow, we obtain

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = \gamma_{xy} = 0$$
$$\gamma_{xz} = 2\lambda\left(\tau_{xz} - kk'\frac{\partial\theta}{\partial\tau_{xz}}\right), \qquad \gamma_{yz} = 2\lambda\left(\tau_{yz} - kk'\frac{\partial\theta}{\partial\tau_{yz}}\right) \tag{2.6}$$

From the expression $\theta = \tan^{-1}(\tau_{yz}/\tau_{xz})$ we find

$$\frac{\partial\theta}{\partial\tau_{xz}} = -\frac{\tau_{yz}}{k^2}, \qquad \frac{\partial\theta}{\partial\tau_{yz}} = \frac{\tau_{xz}}{k^2} \tag{2.7}$$

Consequently, the law of plastic flow may be written in the form

$$\gamma_{xz} = 2\lambda\left(\tau_{xz} + \frac{k'}{k}\tau_{yz}\right), \qquad \gamma_{yz} = 2\lambda\left(\tau_{yz} - \frac{k'}{k}\tau_{xz}\right) \tag{2.8}$$

From (2.8) it follows that

$$\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\left(\tau_{yz} - \frac{k'}{k}\tau_{xz}\right) - \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\left(\tau_{xz} + \frac{k'}{k}\tau_{yz}\right) = 0 \tag{2.9}$$

If we put $u = cyz$, $v = -cxz$ (c = const), then we find from (2.9) that

$$(k\sin\theta - k'\cos\theta)\frac{\partial w}{\partial x} - (k\cos\theta + k'\sin\theta)\frac{\partial w}{\partial y} + cy(k\sin\theta - k'\cos\theta) +$$
$$+ cx(k\cos\theta + k'\sin\theta) = 0 \tag{2.10}$$

It follows from (2.10) that the characteristics of equations (2.4) and (2.10) in the xy-plane coincide, and that along straight lines θ = const the relation

$$w = c\int(y\,dx - x\,dy) + \text{const} \tag{2.11}$$

holds.

Expression (2.11) coincides with the relation of Mandel [6], derived for the warping under torsion of a prismatic rod made from ideally rigid plastic material.

3. We consider the case of a state of plane strain. Direct the z-axis along the axis of an infinite cylinder and assume that none of the components depends upon z. In this case

$$\tau_{xz} = \tau_{yz} = 0, \qquad \varepsilon_z = \varepsilon_{xz} = \varepsilon_{yz} = 0 \tag{3.1}$$

If we limit ourselves to consideration of expression (1.9), then

$$(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 = 4k^2(\theta), \qquad \sigma_z = \tfrac{1}{2}(\sigma_x + \sigma_y) + k(\theta) \tag{3.2}$$

We make a change of variables:

$$\sigma_x = p + k\cos 2\theta, \qquad \sigma_y = p - k\cos 2\theta, \quad \tau_{xy} = k\sin 2\theta \tag{3.3}$$

Upon substitution of (3.3) in the equation of equilibrium we obtain the system

$$\begin{gathered}\frac{\partial p}{\partial x} + k'\cos 2\theta\frac{\partial\theta}{\partial x} - 2k\sin 2\theta\frac{\partial\theta}{\partial x} + k'\sin 2\theta\frac{\partial\theta}{\partial y} + 2k\cos 2\theta\frac{\partial\theta}{\partial y} = 0\\ \frac{\partial p}{\partial y} + k'\sin 2\theta\frac{\partial\theta}{\partial x} + 2k\cos 2\theta\frac{\partial\theta}{\partial x} - k'\cos 2\theta\frac{\partial\theta}{\partial y} + 2k\sin 2\theta\frac{\partial\theta}{\partial y} = 0\end{gathered} \tag{3.4}$$

The characteristics of the system of equations (3.4) have the form

$$\left(\frac{dy}{dx}\right)_{1,2} = \frac{-k'\cos 2\theta - 2k\sin 2\theta \pm \sqrt{k'^2 + 4k^2}}{k'\sin 2\theta + 2k\cos 2\theta} \tag{3.5}$$

It is easy to see that the characteristics (3.5) are mutually orthogonal. In addition, one may obtain relations which are generalizations of the well known Hencky integrals [7]:

$$p \pm F(\theta) = \text{const}, \qquad \left(F(\theta) = \int\sqrt{k'^2 + 4k^2}\,d\theta\right) \tag{3.6}$$

One employs relations (3.6) to formulate a generalization of the Hencky theorem, establishing certain properties of the sliding lines.

We consider the law of plastic flow. We seek the extremum of the work of the stresses for a twisting deformation for a particular element of the body with the additional condition

$$dA = \sigma_x de_x + \sigma_y de_y + 2\tau_{xy}de_{xy} - d\mu\left[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}{}^2 - 4k^2(\theta)\right]$$

The extremum is found from the conditions $\dfrac{\partial A}{\partial\sigma_x} = \dfrac{\partial A}{\partial\sigma_y} = \dfrac{\partial A}{\partial\tau_{xy}} = 0.$ We shall have

$$\begin{gathered}\varepsilon_x = \lambda\left[\sigma_x - \sigma_y - 4kk'\frac{\partial\theta}{\partial\sigma_x}\right]\\ \varepsilon_y = \lambda\left[\sigma_y - \sigma_x - 4kk'\frac{\partial\theta}{\partial\sigma_y}\right], \qquad \varepsilon_{xy} = 4\lambda\left[\tau_{xy} - kk'\frac{\partial\theta}{\partial\tau_{xy}}\right]\end{gathered} \tag{3.7}$$

It follows from (3.3) that $\theta = \tan^{-1}(2\tau_{xy}/\sigma_x - \sigma_y)$, from which we obtain

$$\frac{\partial\theta}{\partial\sigma_x} = -\frac{\sin 2\theta}{4k}, \qquad \frac{\partial\theta}{\partial\sigma_y} = \frac{\sin 2\theta}{4k}, \qquad \frac{\partial\theta}{\partial\tau_{xy}} = \frac{\cos 2\theta}{2k}$$

Finally, the law of plastic flow for anisotropic media takes the form

$$\varepsilon_x = \lambda\left(\sigma_x - \sigma_y + \frac{k'}{k}\tau_{xy}\right), \qquad \varepsilon_y = \lambda\left(\sigma_y - \sigma_x - \frac{k'}{k}\tau_{xy}\right) \tag{3.8}$$
$$\varepsilon_{xy} = 2\lambda\left[2\tau_{xy} - \frac{1}{2}\frac{k'}{k}(\sigma_x - \sigma_y)\right]$$

The expressions (3.8), if λ is eliminated, represent two equations in the two unknowns u and v, which are the components of the displacement velocity. It is easy to convince oneself that the equations are of hyperbolic type and that their characteristics coincide with the characteristics of the stress equations. Along the characteristics the elongation is zero, and so the relations of Geiringer hold [8].

We note that the form of the plasticity conditions (3.2) is similar to the Tresca plasticity conditions, but because of anisotropy, the sliding lines do not coincide with the lines of maximum shear stress. It should be noted also that conditions (1.4) may be realized to some extent in the xy plane, and so in the general case, it is necessary to provide for the possibility of conjugate solutions.

4. Finally, we consider the case of a state of plane stress. We suppose that $\sigma_3 = \sigma_z = 0$, $\tau_{xz} = \tau_{yz} = 0$. |+ l In Table (1.2) the relations $n_3 = 1$, $n_1 = n_2 = l_3 = m_3 = 0$ hold. On these assumptions the quantities k_3 and s_3 are constants.

The plasticity conditions are described by a hexagon as in Fig. 3. It follows from (1.4) that in the general case the plasticity conditions may be written in the form

$$A(\theta)\,\sigma_1 + B(\theta)\,\sigma_2 = 1 \tag{4.1}$$

Passing over to the Cartesian components we obtain

$$\sigma_x = p + [C_1(\theta) - C_2(\theta)\,p]\cos^2\theta \qquad \left(C_1 = \frac{1}{A}\right)$$
$$\sigma_y = p + [C_1(\theta) - C_2(\theta)\,p]\sin^2\theta \qquad \left(C_2 = \frac{A+B}{A}\right) \tag{4.2}$$
$$\tau_{xy} = -[C_1(\theta) - C_2(\theta)\,p]\sin\theta\cos\theta$$

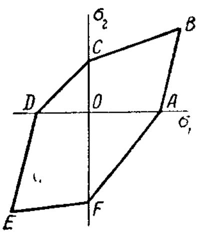

Fig. 3.

For the sides of the hexagon AB, BC, DE, and EF in Fig. 3.we shall have $k_2 = \kappa k_1$, where κ is a constant equal to k_3 or s_3.

The plasticity conditions (4.1) may be rewritten as

$$AB(\sigma_x^2+\sigma_y^2)+(A^2+B^2)\,\sigma_x\sigma_y-(A+B)(\sigma_x+\sigma_y)-[(A-B)\tau_{xy}]^2+1=0$$

Upon substitution of expressions (4.2) into the equation of equilibrium we obtain

$$\frac{\partial p}{\partial x}(1-C_2\cos^2\theta)+\frac{1}{2}C_2\sin 2\theta\frac{\partial p}{\partial y}+(D\sin 2\theta+D'\cos^2\theta)\frac{\partial\theta}{\partial x}-\Delta\frac{\partial\theta}{\partial y}=0$$

$$\frac{\partial p}{\partial y}(1-C_2\sin^2\theta)+\frac{1}{2}C_2\sin 2\theta\frac{\partial p}{\partial x}+(D\sin 2\theta+D'\sin^2\theta)\frac{\partial\theta}{\partial y}-\Delta\frac{\partial\theta}{\partial x}=0$$

Here

$$D=C_1-C_2p,\qquad D'=\frac{dC_1}{d\theta}-p\frac{dC_2}{d\theta},\quad \Delta=D\cos 2\theta+\frac{1}{2}D'\sin 2\theta \tag{4.4}$$

The characteristic equations for the system (4.4) are written in the form

$$\left(\frac{dy}{dx}\right)_{1,2}=\frac{D'\cos 2\theta-D(2-C_2)\sin 2\theta\pm\sqrt{4D^2(2-C_2)+D'^2}}{2(-DC_2\cos^2\theta+D\cos 2\theta+{}^1/_2D'\sin 2\theta)} \tag{4.5}$$

Along the characteristics the relations

$$p+F(\theta)=\text{const},\qquad F(\theta)=\int\frac{D'\pm\sqrt{4D'(1-C_2)+D'^2}}{2(1-C_2)} \tag{4.6}$$

hold.

Let us determine the law of plastic flow. Upon making use of relations (4.4) as in a plastic potential we find that

$$\varepsilon_x=\lambda\{2AB\sigma_x+(A^2+B^2)\sigma_y-(A+B)-[(A'B+AB')(\sigma_x^2+\sigma_y^2)+$$
$$+2(AA'+BB')\sigma_x\sigma_y-(A'+B')(\sigma_x+\sigma_y)-2(A-B)(A'-B')\tau_{xy}^2]\tau_{xy}^\circ\}$$
$$\varepsilon_y=\lambda\{2AB\sigma_y+(A^2+B^2)\sigma_x-(A+B)-[(A'B+AB')(\sigma_x^2+\sigma_y^2)+$$
$$+2(AA'+BB')\sigma_x\sigma_y-(A'+B')(\sigma_x+\sigma_y)-2(A-B)(A'-B')\tau_{xy}^2]\tau_{xy}^\circ\}$$
$$2\varepsilon_{xy}=\lambda\{-2(A-B)^2\tau_{xy}+[(A'B+AB')(\sigma_x^2+\sigma_y^2)+2(AA'+BB')\sigma_x\sigma_y-$$
$$-(A'+B')(\sigma_x+\sigma_y)-2(A-B)(A'-B')\tau_{xy}^2]\sigma^\circ$$

$$\theta=\frac{1}{2}\operatorname{arc\,tg}\frac{2\tau_{xy}}{\sigma_x-\sigma_y},\quad \tau_{xy}^\circ=\frac{\tau_{xy}}{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2},\quad \sigma^\circ=\frac{\sigma_x-\sigma_y}{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}$$

It is easy to convenice oneself that the equations determining the field of velocity belong to a hyperbolic type and that their characteristics coincide with the characteristics of (4.5).

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